The classical limit of so(3) compact quantum systems

1992 J. Phys. A: Math. Gen. 25 1685

(http://iopscience.iop.org/0305-4470/25/6/023)

J. Phys. A: Math. Gen. 25 (1992) 1685–1702. Printed in the UK

The classical limit of so(3) compact quantum systems

R J B Fawcett

School of Mathematics, Queensland University of Technology, Gardens Point Campus, GPO Box 2434, Brisbane, QLD, 4001, Australia

Received 2 August 1991, in final form 18 November 1991

Abstract. In a recent paper by Fawcett and Bracken, the classical limit of ordinary non-compact quantum systems is considered in terms of the contraction of the underlying kinematical Lie algebra (the Weyl–Heisenberg algebra) and its representations to the Abelian Lie algebra of the same dimension and its representations. In this paper, some of those ideas are adapted to discuss in similar terms, the classical limit of compact quantum systems whose underlying kinematical algebras are of the special orthogonal type. The classical dynamical system that results from the classical limit of a compact quantum mechanical system will be called a 'compact classical system'. Poisson brackets for such 'compact classical systems' have already been given in the literature (Grossmann and Peres), and the recovery of the Poisson bracket for so(3) compact classical systems is demonstrated in terms of the contraction limit.

1. Introduction

In a recent paper [1], the classical limit of quantum mechanics was considered in terms of the contraction of the underlying kinematical algebra, the non-compact Weyl–Heisenberg Lie algebra w_n (and its representations) to the non-compact Abelian Lie algebra t_{2n+1} of the same dimension (and its representations). To accommodate this purpose, a new definition of the contraction of a Lie algebra and its matrix representations by the method of sequences of representations was presented. A detailed formulation with examples of the $n = 1$ case was given, although an important step involving the interchanging of two limiting processes remains to be proved. The formulation of the $n > 1$ cases is a straightforward generalization, but similar obstacles as yet stand in the way of a complete proof.

In this paper, the discussion is extended to the idea of the classical limit of a compact quantum system. A compact quantum system is a quantum mechanical system whose underlying kinematical algebra is a compact Lie algebra. The idea of a compact quantum system based on the representation theory of $\mathrm{so}(n+2)$ arose from work by Barut, Bracken and Thacker [2–5]. These ideas were extended to a general formalism of compact quantum systems based on the special orthogonal and unitary algebras [6] and the compact symplectic algebras [7]. It is that formalism for the compact quantum systems based on so(3) which is used here.

The notion of a 'compact classical system', a classical dynamical system obtained as the classical limit of a compact quantum system, is not entirely new. Grossmann and Peres [8] postulated a Poisson bracket obtained from compact Lie algebras, as an analogue to the way the ordinary Poisson bracket of classical Hamiltonian mechanics is obtained in the classical limit from the commutator of quantum mechanics. The compact Poisson bracket of Grossmann and Peres [8] will be derived later in the case of so(3) compact quantum systems.

A detailed discussion of the properties of $\mathrm{so}(n)$ compact classical systems will not be presented here. Some of the essential features of such systems have been considered elsewhere [9], however the important points will be mentioned. As the formulation of the classical limit of an so(3) compact quantum system is in many ways similar to that of a w_1 non-compact quantum system [1], the discussion in places will be somewhat abbreviated.

Throughout this paper, t_n is used to denote the Abelian Lie algebra with n generators.

2. Contraction of representations of so(3) to representations of t_3

Let $\bar{a}, a$ be a pair of boson creation and annihilation operators on a Hilbert space $\mathcal{H}$ which contains vectors $\xi_r = \left(1/\sqrt{r!}\right) \bar{a}^r \xi_0$ for $r = 0, 1, \ldots$, such that

$$a\xi_0 = 0 \qquad a\xi_r = \sqrt{r}\xi_{r-1} \qquad \bar{a}\xi_r = \sqrt{r+1}\xi_{r+1}. \tag{1}$$

Let $N = \bar{a}a$ be the number operator.

The Lie algebra so(3) has three generators X_1, X_2, X_3, which satisfy the bracket relations

$$[X_1, X_2] = \mathrm{i}\hbar X_3 \qquad [X_1, X_3] = \frac{-\mathrm{i}\lambda}{l\kappa} X_2 \qquad [X_2, X_3] = \frac{\mathrm{i}\kappa}{l\lambda} X_1. \tag{2}$$

In compact quantum mechanics [6], the generators X_1, X_2 are associated with compact position and momentum operators, respectively, and $\hbar$ is the modified Planck's constant. In this paper, the boson realization [6,10]

$$\begin{aligned} Q_l &= \pi_l(X_1) = (\lambda/2\sqrt{l})[a + \bar{a}(2l\mathbb{I} - N)] \\ P_l &= \pi_l(X_2) = (-\mathrm{i}\kappa/2\sqrt{l})[a - \bar{a}(2l\mathbb{I} - N)] \\ J_l &= \pi_l(X_3) = -(1/l)(N - l\mathbb{I}) \end{aligned} \tag{3}$$

on the span of the ξ_r will be used, where λ and κ are fixed length and momentum scales respectively satisfying $\lambda\kappa = \hbar$ and $\mathbb{I}$ is the identity operator. This realization is conventionally labelled (l). In the cases where $2l$ is a non-negative integer, the action of the generators Q_l, P_l, J_l leaves the finite-dimensional subspace

$$S_l = \mathrm{span}\{\xi_0, \xi_1, \ldots, \xi_{2l}\} \tag{4}$$

invariant. Cases where $2l$ is not a non-negative integer will not be further considered in this paper. If it is assumed that the vectors ξ_r are orthonormal with respect to some inner product $(\cdot, \cdot)$ on $\mathcal{H}$, the so(3) boson realization (3) will not be Hermitian with respect to $(\cdot, \cdot)$. Therefore, introduce rescaled vectors

$$\xi_{l,r} = \sqrt{\frac{(2l)!}{r!(2l-r)!}} \bar{a}^r \xi_0 \qquad r = 0, 1, \ldots, 2l \tag{5}$$

and a new inner product $(\cdot,\cdot)_l$ with respect to which the $\xi_{l,r}$ are orthonormal. The operators a and $\bar{a}(2l\mathbb{I}-N)$ are now Hermitian conjugate pairs with respect to $(\cdot,\cdot)_l$ and $\mathcal{S}_l$ is the span of the $\xi_{l,r}$. Using these new vectors (5), matrix elements for the generators can be determined from (3) by the equations

$$\begin{aligned} Q_l\xi_{l,r} &= (\lambda/2\sqrt{l})[\sqrt{r(2l+1-r)}\xi_{l,r-1}+\sqrt{(r+1)(2l-r)}\xi_{l,r+1}] \\ P_l\xi_{l,r} &= (-\mathrm{i}\kappa/2\sqrt{l})[\sqrt{r(2l+1-r)}\xi_{l,r-1}-\sqrt{(r+1)(2l-r)}\xi_{l,r+1}] \\ J_l\xi_{l,r} &= -(1/l)(r-l)\xi_{l,r} \end{aligned} \tag{6}$$

for $r=0,1,\ldots,2l$. The matrices Q_l, P_l, J_l satisfy the commutation relations

$$[Q_l,P_l]=\mathrm{i}\hbar J_l \qquad [Q_l,J_l]=\frac{-\mathrm{i}\lambda}{l\kappa}P_l \qquad [P_l,J_l]=\frac{\mathrm{i}\kappa}{l\lambda}Q_l \tag{7}$$

and the Casimir relationship satisfied by them is

$$({Q_l}^2/\lambda^2+{P_l}^2/\kappa^2)/l+{J_l}^2=(1+1/l)\mathbb{I}. \tag{8}$$

The Lie algebra so(3) is contracted to t_3 by setting $X_1^\epsilon=\epsilon X_1$, $X_2^\epsilon=\epsilon X_2$, $X_3^\epsilon=\epsilon X_3$, where ϵ is the contraction parameter. The bracket relations satisfied by the contracting generators are then

$$[X_1^\epsilon,X_2^\epsilon]=\mathrm{i}\epsilon\hbar X_3^\epsilon \qquad [X_1^\epsilon,X_3^\epsilon]=\frac{-\mathrm{i}\epsilon\lambda}{l\kappa}X_2^\epsilon \qquad [X_2^\epsilon,X_3^\epsilon]=\frac{\mathrm{i}\epsilon\kappa}{l\lambda}X_1^\epsilon \tag{9}$$

and these all formally vanish in the contraction limit, so that the bracket relations reduce to those defining t_3, that is $[Y_i,Y_j]=0$ for $i,j=1,2,3$, where the Y_i denote the generators of the contracted algebra.

Theorem. Choose a dimensionless constant ζ such that $0\leqslant\zeta\leqslant 2l$. Then there is a sequence of $(2ml+1)$-dimensional matrix representations $\{\pi_{(m)l}\}_{m=1}^{\infty}$ of so(3) labelled (ml) in the usual way, acting on $(2ml+1)$-dimensional subspaces $\mathcal{S}_{(m)l}$ of Hilbert spaces $\mathcal{H}_{(m)l}$, whose contraction limit is a representation $\pi_{(\infty)l}$ of t_3 acting on a subspace $\mathcal{S}_{(\infty)l}$ of $\mathcal{H}_{(\infty)l}$. This representation is equivalent to a direct integral of irreducible one-dimensional Hermitian representations with

$$\begin{aligned} Q_{(\infty)l} &= \lambda\int_0^{2\pi}\oplus[\sqrt{\zeta(2-\zeta/l)}\cos\theta]\mathrm{d}\theta \\ P_{(\infty)l} &= \kappa\int_0^{2\pi}\oplus[\sqrt{\zeta(2-\zeta/l)}\sin\theta]\mathrm{d}\theta \\ J_{(\infty)l} &= \int_0^{2\pi}\oplus[1-\zeta/l]\mathrm{d}\theta \end{aligned} \tag{10}$$

and

$$({Q_{(\infty)l}}^2/\lambda^2+{P_{(\infty)l}}^2/\kappa^2)/l+{J_{(\infty)l}}^2=\mathbb{I}_{(\infty)l} \tag{11}$$

where

$$Q_{(\infty)l}=\pi_{(\infty)l}(Y_1) \qquad P_{(\infty)l}=\pi_{(\infty)l}(Y_2) \qquad J_{(\infty)l}=\pi_{(\infty)l}(Y_3) \tag{12}$$

and $\mathbb{I}_{(\infty)l}$ is the identity operator on $\mathcal{H}_{(\infty)l}$.

Proof. Let $[\![\alpha]\!]$ denote the integer part of the real number α and set $M_{(m)l} = [\![m\zeta]\!]$. Consider a sequence of $(2ml+1)$-dimensional representations $\pi_{(m)l}$ of so(3), labelled by $m = 1, 2, \ldots$, where for each m,

$$\begin{aligned}
\pi_{(m)l}(X_1) &= Q_l = (\lambda/2\sqrt{l})[a + \bar{a}(2ml\mathbb{I} - N)] \\
\pi_{(m)l}(X_2) &= P_l = (-\mathrm{i}\kappa/2\sqrt{l})[a - \bar{a}(2ml\mathbb{I} - N)] \\
\pi_{(m)l}(X_3) &= J_l = -(1/l)(N - ml\mathbb{I})
\end{aligned} \tag{13}$$

Q_l, P_l, J_l still obeying (7), with matrix elements on the basis vectors $\{\xi_{ml,0}, \xi_{ml,1}, \ldots, \xi_{ml,2ml}\}$ being given by

$$\begin{aligned}
Q_l\xi_{ml,r} &= (\lambda/2\sqrt{l})[\sqrt{r(2ml+1-r)}\xi_{ml,r-1} + \sqrt{(r+1)(2ml-r)}\xi_{ml,r+1}] \\
P_l\xi_{ml,r} &= (-\mathrm{i}\kappa/2\sqrt{l})[\sqrt{r(2ml+1-r)}\xi_{ml,r-1} - \sqrt{(r+1)(2ml-r)}\xi_{ml,r+1}] \\
J_l\xi_{ml,r} &= -(1/l)(r - ml)\xi_{ml,r}
\end{aligned} \tag{14}$$

for $r = 0, 1, \ldots, 2ml$. However, for the purposes of the contraction process, $\pi_{(m)l}$ will be given on $\mathcal{S}_{(m)l}$ in terms of a re-ordered basis

$$\{\psi_{(m)r} | \psi_{(m)r} = \xi_{ml,r+M_{(m)l}}, r = -M_{(m)l}, \ldots, 2ml - M_{(m)l}\}. \tag{15}$$

For each $m \geqslant 1$, the contraction parameter ϵ_m is taken to have the value $1/m$, and set

$$\begin{aligned}
Q_{(m)l} &= \epsilon_m Q_l = \pi_{(m)l}(X_1^\epsilon) = \epsilon_m \pi_{(m)l}(X_1) \\
P_{(m)l} &= \epsilon_m P_l = \pi_{(m)l}(X_2^\epsilon) = \epsilon_m \pi_{(m)l}(X_2) \\
J_{(m)l} &= \epsilon_m J_l = \pi_{(m)l}(X_3^\epsilon) = \epsilon_m \pi_{(m)l}(X_3).
\end{aligned} \tag{16}$$

On the vectors $\psi_{(m)r}$,

$$\begin{aligned}
Q_{(m)l}\psi_{(m)r} &= (\lambda/2m\sqrt{l}) \Big[\sqrt{(M_{(m)l} + r)(2ml + 1 - M_{(m)l} - r)}\,\psi_{(m)r-1} \\
&\quad + \sqrt{(M_{(m)l} + r + 1)(2ml - M_{(m)l} - r)}\,\psi_{(m)r+1}\Big] \\
P_{(m)l}\psi_{(m)r} &= (-\mathrm{i}\kappa/2m\sqrt{l}) \Big[\sqrt{(M_{(m)l} + r)(2ml + 1 - M_{(m)l} - r)}\,\psi_{(m)r-1} \\
&\quad - \sqrt{(M_{(m)l} + r + 1)(2ml - M_{(m)l} - r)}\,\psi_{(m)r+1}\Big] \\
J_{(m)l}\psi_{(m)r} &= -(1/ml)(M_{(m)l} + r - ml)\psi_{(m)r}
\end{aligned} \tag{17}$$

and

$$[({Q_{(m)l}}^2/\lambda^2 + {P_{(m)l}}^2/\kappa^2)/l + {J_{(m)l}}^2]\psi_{(m)r} = (1 + 1/ml)\psi_{(m)r} \tag{18}$$

for all integers r such that $-M_{(m)l} \leqslant r \leqslant 2ml - M_{(m)l}$. In the contraction limit,

$$Q_{(\infty)l}\psi_{(\infty)r} = (\lambda/2)\sqrt{\zeta(2-\zeta/l)}[\psi_{(\infty)r-1} + \psi_{(\infty)r+1}]$$
$$P_{(\infty)l}\psi_{(\infty)r} = (-\mathrm{i}\kappa/2)\sqrt{\zeta(2-\zeta/l)}[\psi_{(\infty)r-1} - \psi_{(\infty)r+1}]$$
$$J_{(\infty)l}\psi_{(\infty)r} = (1-\zeta/l)\psi_{(\infty)r} \tag{19}$$

for all integers r. Minor adjustments, here and later, have to be made in the cases $\zeta = 0$ where $r \geqslant 0$ in (19), and $\zeta = 2l$ where $r \leqslant 0$ in (19). It is easily checked that

$$[Q_{(\infty)l}, P_{(\infty)l}]\psi_{(\infty)r} = [Q_{(\infty)l}, J_{(\infty)l}]\psi_{(\infty)r} = [P_{(\infty)l}, J_{(\infty)l}]\psi_{(\infty)r} = 0 \tag{20}$$

and

$$[(Q_{(\infty)l}{}^2/\lambda^2 + P_{(\infty)l}{}^2/\kappa^2)/l + J_{(\infty)l}{}^2]\psi_{(\infty)r} = \psi_{(\infty)r} \tag{21}$$

for all integers r, and thus (11) holds for the representation of t_3 obtained here.

It can be seen as follows that the representation of t_3 obtained here is a direct integral of one-dimensional Hermitian representations.

Let $(\cdot,\cdot)_{(\infty)l}$ be the inner product of the Hilbert space $\mathcal{H}_{(\infty)l}$ with respect to which the $\psi_{(\infty)r}$ are orthonormal for all integers r, $\mathcal{S}_{(\infty)l}$ the linear span of the $\psi_{(\infty)r}$, and let $\mathbb{I}_{(\infty)l}$ be the identity operator on $\mathcal{H}_{(\infty)l}$. Also introduce the dual space $\mathcal{T}_{(\infty)l}$ of $\mathcal{S}_{(\infty)l}$, whose elements (linear functionals on $\mathcal{S}_{(\infty)l}$) are associated with all formal vectors of the form $\sum_{r=-\infty}^{\infty} c_r\psi_{(\infty)r}$, where the c_r are arbitrary complex numbers. The commuting operators $Q_{(\infty)l}$, $P_{(\infty)l}$ $J_{(\infty)l}$ have common generalized eigenvectors in $\mathcal{T}_{(\infty)l}$. They have the form

$$\Phi_{(\infty)l}(\zeta,\theta) = \frac{1}{\sqrt{2\pi}}\sum_{r=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}r\theta}\psi_{(\infty)r} \qquad 0 \leqslant \theta < 2\pi \tag{22}$$

and they can be seen from (19) to satisfy

$$Q_{(\infty)l}\Phi_{(\infty)l}(\zeta,\theta) = \lambda\sqrt{\zeta(2-\zeta/l)}\cos\theta\,\Phi_{(\infty)l}(\zeta,\theta)$$
$$P_{(\infty)l}\Phi_{(\infty)l}(\zeta,\theta) = \kappa\sqrt{\zeta(2-\zeta/l)}\sin\theta\,\Phi_{(\infty)l}(\zeta,\theta)$$
$$J_{(\infty)l}\Phi_{(\infty)l}(\zeta,\theta) = (1-\zeta/l)\Phi_{(\infty)l}(\zeta,\theta) \tag{23}$$

in the sense that

$$(\Phi_{(\infty)l}(\zeta,\theta), Q_{(\infty)l}\psi_{(\infty)r})_{(\infty)l} = \lambda\sqrt{\zeta(2-\zeta/l)}\cos\theta(\Phi_{(\infty)l}(\zeta,\theta),\psi_{(\infty)r})_{(\infty)l}$$
$$(\Phi_{(\infty)l}(\zeta,\theta), P_{(\infty)l}\psi_{(\infty)r})_{(\infty)l} = \kappa\sqrt{\zeta(2-\zeta/l)}\sin\theta(\Phi_{(\infty)l}(\zeta,\theta),\psi_{(\infty)r})_{(\infty)l}$$
$$(\Phi_{(\infty)l}(\zeta,\theta), J_{(\infty)l}\psi_{(\infty)r})_{(\infty)l} = (1-\zeta/l)(\Phi_{(\infty)l}(\zeta,\theta),\psi_{(\infty)r})_{(\infty)l} \tag{24}$$

for all integers r. These generalized eigenvectors have a 'delta function normalization',

$$(\Phi_{(\infty)l}(\zeta,\theta), \Phi_{(\infty)l}(\zeta,\theta'))_{(\infty)l} = \delta(\theta'-\theta) \tag{25}$$

for $0 \leqslant \theta, \theta' < 2\pi$. It can be shown that the vectors $\Phi_{(\infty)l}(\zeta,\theta)$ form a (continuous) basis for the Hilbert space $\mathcal{H}_{(\infty)l}$ spanned by the vectors $\psi_{(\infty)r}$; in particular,

$$\psi_{(\infty)r} = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}r\theta}\Phi_{(\infty)l}(\zeta,\theta)\,\mathrm{d}\theta. \tag{26}$$

This completes the proof of the theorem. □

The reducible (direct integral) representation of t_3 obtained here in the contraction limit is effectively parameterized by the real number ζ. Direct sum combinations of representations of this type can be found [7] along the lines of the method used for the analogous non-compact result [1]. As in that case, it appears that the maximum extent of the direct sum combination here is a countably infinite direct sum over ζ values together with a direct integral over θ values.

3. The classical limit of an so(3) compact quantum system with a single degree of freedom

Consider now a compact quantum system with a single degree of freedom, kinematical algebra so(3), and a time-independent Hamiltonian operator H_l. The representation of so(3) is spanned by operators Q_l, P_l, J_l as in (3), acting on $\mathcal{S}_l$.

Suppose H_l can be written as a polynomial in Q_l, P_l and J_l of the form

$$H_l(Q_l, P_l, J_l) = \sum_{m_1,m_2,m_3} a_{m_1m_2m_3}\tfrac{1}{6}(Q_l{}^{m_1}P_l{}^{m_2}J_l{}^{m_3} + \cdots + J_l{}^{m_3}P_l{}^{m_2}Q_l{}^{m_1}) \tag{27}$$

where the $a_{m_1m_2m_3}$ are real constants of the appropriate dimensions, and all six permutations of the Q_l, P_l and J_l are included. The sum in (27) and following equations is to be taken over non-negative integers m_1, m_2, m_3 such that $m_1 + m_2 + m_3 \leqslant N$ for some fixed positive integer N. In this form, the Hamiltonian operator is Hermitian with respect to $(\cdot,\cdot)_l$ and due to the finite dimension of $\mathcal{S}_l$, H_l is self-adjoint with respect to $(\cdot,\cdot)_l$ on $\mathcal{S}_l$.

The time dependence of the quantum operators in the Heisenberg picture is taken to be given [7] by the operator differential equations

$$\begin{aligned}
\dot{Q}_l = \frac{1}{\mathrm{i}\hbar}[Q_l, H_l] = \frac{1}{\mathrm{i}\hbar}\sum_{m_1,m_2,m_3} a_{m_1m_2m_3}\frac{1}{6}\bigg[\frac{-\mathrm{i}\lambda}{l\kappa}(Q_l(t)^{m_1}P_l(t)^{m_2}[J_l(t)^{m_3-1}P_l(t)\\
+\cdots+P_l(t)J_l(t)^{m_3-1}]+\cdots) + \mathrm{i}\hbar(Q_l(t)^{m_1}[P_l(t)^{m_2-1}J_l(t)+\cdots\\
+J_l(t)P_l(t)^{m_2-1}]J_l(t)^{m_3}+\cdots)\bigg]
\end{aligned}$$

$$\begin{aligned}
\dot{P}_l = \frac{1}{\mathrm{i}\hbar}[P_l, H_l] = \frac{1}{\mathrm{i}\hbar}\sum_{m_1,m_2,m_3} a_{m_1m_2m_3}\frac{1}{6}\bigg[\frac{\mathrm{i}\kappa}{l\lambda}(Q_l(t)^{m_1}P_l(t)^{m_2}[J_l(t)^{m_3-1}Q_l(t)\\
+\cdots+Q_l(t)J_l(t)^{m_3-1}]+\cdots) - \mathrm{i}\hbar([Q_l(t)^{m_1-1}J_l(t)+\cdots\\
+J_l(t)Q_l(t)^{m_1-1}]P_l(t)^{m_2}J_l(t)^{m_3}+\cdots)\bigg]
\end{aligned}$$

$$\begin{aligned}
\dot{J}_l = \frac{1}{\mathrm{i}\hbar}[J_l, H_l] = \frac{1}{\mathrm{i}\hbar}\sum_{m_1,m_2,m_3} a_{m_1m_2m_3}\frac{1}{6}\bigg[\frac{-\mathrm{i}\kappa}{l\lambda}(Q_l(t)^{m_1}[P_l(t)^{m_2-1}Q_l(t)+\cdots\\
+Q_l(t)P_l(t)^{m_2-1}]J_l(t)^{m_3}+\cdots) + \frac{\mathrm{i}\lambda}{l\kappa}([Q_l(t)^{m_1-1}P_l(t)+\cdots\\
+P_l(t)Q_l(t)^{m_1-1}]P_l(t)^{m_2}J_l(t)^{m_3}+\cdots)\bigg]
\end{aligned} \tag{28}$$

where all the permutations of $Q_l^{m_1} P_l^{m_2} J_l^{m_3}$ are duly included. Formal solutions to these equations have the form

$$Q_l(t) = U_l(t)^\dagger Q_l(0) U_l(t) \qquad P_l(t) = U_l(t)^\dagger P_l(0) U_l(t)$$

$$J_l(t) = U_l(t)^\dagger J_l(0) U_l(t) \qquad U_l(t) = \exp[H_l(Q_l(0), P_l(0), J_l(0))t/\mathrm{i}\hbar] \tag{29}$$

where $U_l(t)$ is the unitary evolution operator.

The corresponding compact classical Hamiltonian system has a Hamiltonian function

$$H(q,p,u) = \sum_{m_1,m_2,m_3} a_{m_1 m_2 m_3} q^{m_1} p^{m_2} u^{m_3} \tag{30}$$

where q, p, u are the compact classical variables corresponding to Q, P, J respectively, and dynamical equations

$$\begin{aligned} \dot{q} &= -\frac{1}{l\kappa^2} p \frac{\partial H}{\partial u} + u \frac{\partial H}{\partial p} \\ \dot{p} &= \frac{1}{l\lambda^2} q \frac{\partial H}{\partial u} - u \frac{\partial H}{\partial q} \\ \dot{u} &= -\frac{1}{l\lambda^2} q \frac{\partial H}{\partial p} + \frac{1}{l\kappa^2} p \frac{\partial H}{\partial q}. \end{aligned} \tag{31}$$

These equations are obtained by analogy from the expansions of the commutators in (28), and the implicit presence of $\hbar$ in these equations is unavoidable. The variable u has been used here in preference to j because j, like l, is commonly used as a label for so(3) representations. There is no compact classical variable corresponding to the evolution operator $U_l(t)$. There are immediately two constants of the motion obtainable from (31). The first is the compact energy integral $I_1 = H(q,p,u)$. The second is

$$I_2 = (q^2/\lambda^2 + p^2/\kappa^2)/l + u^2 \tag{32}$$

which from (11) will be taken to have the value of unity. The equation (32) constrains the compact classical motion to a (closed and bounded) ellipsoid of revolution in the phase space of the variables q, p, u, and this contraint leads to some interesting features of such systems [9].

Let (q_0, p_0, u_0) be an initial condition for a compact classical trajectory $(q(t), p(t), u(t))$ of (31) and set $q_0 = \lambda\sqrt{\zeta(2-\zeta/l)}\cos\theta$, $p_0 = \kappa\sqrt{\zeta(2-\zeta/l)}\sin\theta$, $u_0 = (1-\zeta/l)$ for $\zeta(2-\zeta/l) = [(q_0/\lambda)^2 + (p_0/\kappa)^2] \geqslant 0$, $0 \leqslant \zeta \leqslant 2l$, $0 \leqslant \theta < 2\pi$, $|u_0| \leqslant 1$ and $2l$ a positive integer. Here, λ, κ are the scales introduced in the previous section. Note that the scale of $|q_0 p_0|$ can be made macroscopic, or large compared with $\hbar$, by choosing ζ and l sufficiently large.

Consider now the contraction of so(3) to t_3 given in the previous section, where the number states ξ_r are eigenvectors of $\bar{a}(0)a(0)$, and $a(0)$ and $\bar{a}(0)$ are related to $Q_l(0)$, $P_l(0)$ and $J_l(0)$ as in (13). Again set $Q_{(m)l} = \epsilon_m Q_l$, $P_{(m)l} = \epsilon_m P_l$,

$J_l = \epsilon_m J_l$, and let $Q_{(m)l}$, $P_{(m)l}$, $J_{(m)l}$ have the representation (16), at time $t = 0$. Then from (17),

$$Q_{(m)l}(0)\psi_{(m)r} = (\lambda/2m\sqrt{l})[\sqrt{(M_{(m)l}+r)(2ml+1-M_{(m)l}-r)}\psi_{(m)r-1} + \sqrt{(M_{(m)l}+r+1)(2ml-M_{(m)l}-r)}\psi_{(m)r+1}]$$

$$P_{(m)l}(0)\psi_{(m)r} = (-\mathrm{i}\kappa/2m\sqrt{l})[\sqrt{(M_{(m)l}+r)(2ml+1-M_{(m)l}-r)}\psi_{(m)r-1} - \sqrt{(M_{(m)l}+r+1)(2ml-M_{(m)l}-r)}\,\psi_{(m)r+1}]$$

$$J_{(m)l}(0)\psi_{(m)r} = -(1/ml)(M_{(m)l}+r-ml)\psi_{(m)r} \tag{33}$$

and

$$\{[(Q_{(m)l}(0)/\lambda)^2 + (P_{(m)l}(0)/\kappa)^2]/l + J_{(m)l}(0)^2\}\psi_{(m)r} = (1+1/ml)\psi_{(m)r} \tag{34}$$

for $-M_{(m)l} \leqslant r \leqslant 2ml - M_{(m)l}$. The action of $Q_{(\infty)l}(0)$, $P_{(\infty)l}(0)$ and $J_{(\infty)l}(0)$, the operators obtained in the contraction limit, on the vectors $\psi_{(\infty)r}$ for all integers r is given by (19). On the generalized eigenvectors $\Phi_{(\infty)l}(\zeta,\theta)$,

$$Q_{(\infty)l}(0)\Phi_{(\infty)l}(\zeta,\theta) = q_0\Phi_{(\infty)l}(\zeta,\theta)$$

$$P_{(\infty)l}(0)\Phi_{(\infty)l}(\zeta,\theta) = p_0\Phi_{(\infty)l}(\zeta,\theta)$$

$$J_{(\infty)l}(0)\Phi_{(\infty)l}(\zeta,\theta) = u_0\Phi_{(\infty)l}(\zeta,\theta) \tag{35}$$

in the sense described in (24).

The time evolution of the contracting operators $Q_{(m)l}(t)$, $P_{(m)l}(t)$, $J_{(m)l}(t)$ is taken to be determined by the operator differential equations

$$\dot{Q}_{(m)l}(t) = \frac{1}{\mathrm{i}\epsilon_m\hbar}[Q_{(m)l}(t), H_{(m)l}]$$

$$\dot{P}_{(m)l}(t) = \frac{1}{\mathrm{i}\epsilon_m\hbar}[P_{(m)l}(t), H_{(m)l}]$$

$$\dot{J}_{(m)l}(t) = \frac{1}{\mathrm{i}\epsilon_m\hbar}[J_{(m)l}(t), H_{(m)l}] \tag{36}$$

where

$$H_{(m)l} \equiv H_{(m)l}(Q_{(m)l}(0), P_{(m)l}(0), J_{(m)l}(0)) = \sum_{m_1,m_2,m_3} a_{m_1m_2m_3}\tfrac{1}{6}(Q_{(m)l}(0)^{m_1}P_{(m)l}(0)^{m_2}J_{(m)l}(0)^{m_3} + \cdots + J_{(m)l}(0)^{m_3}P_{(m)l}(0)^{m_2}Q_{(m)l}(0)^{m_1}). \tag{37}$$

The expansions of (36), not included here, for the form of the Hamiltonian (37), are similar to those of (28), from which (31) was deduced. Formal solutions of (36) in terms of a unitary evolution operator $U_{(m)l}(t)$ can be given in the form

$$\begin{aligned}
Q_{(m)l}(t) &= U_{(m)l}(t)^{\dagger} Q_{(m)l}(0) U_{(m)l}(t) \\
P_{(m)l}(t) &= U_{(m)l}(t)^{\dagger} P_{(m)l}(0) U_{(m)l}(t) \\
J_{(m)l}(t) &= U_{(m)l}(t)^{\dagger} J_{(m)l}(0) U_{(m)l}(t) \\
U_{(m)l}(t) &= \exp[H_{(m)l} t / i\epsilon_m \hbar].
\end{aligned} \tag{38}$$

The power of ϵ_m appearing in (36) and (38) is determined by the consideration that (36) should in general not diverge or vanish as $\epsilon_m \to 0$. This power of ϵ_m is not the same as the power appearing in the non-compact case [1].

Let $(\cdot, \cdot)_{(m)l}$ be an inner product on the Hilbert space $\mathcal{H}_{(m)l}$ with respect to which the $\psi_{(m)r}$ are orthonormal for integers r, $-M_{(m)l} \leqslant r \leqslant 2ml - M_{(m)l}$.

Let $S_{(m)l}$ denote the span of the vectors $\psi_{(m)r}$, $-M_{(m)l} \leqslant r \leqslant 2ml - M_{(m)l}$ and let $\mathcal{T}_{(m)l}$ denote its dual space which is associated with all formal vectors of the form

$$\sum_{r=-M_{(m)l}}^{2ml-M_{(m)l}} c_r \psi_{(m)r}$$

where the c_r are complex numbers. Let $\mathbb{I}_{(m)l}$ denote the identity operator on $\mathcal{H}_{(m)l}$ and set

$$\Phi_{(m)l}(\zeta, \theta) = \frac{1}{\sqrt{2\pi}} \sum_{r=-M_{(m)l}}^{2ml-M_{(m)l}} \mathrm{e}^{\mathrm{i}r\theta} \psi_{(m)r}. \tag{39}$$

From the definitions (39) and (22) of $\Phi_{(m)l}(\zeta, \theta)$ and $\Phi_{(\infty)l}(\zeta, \theta)$ respectively, and the definition of the contraction of a sequence of representations, it follows that

$$\lim_{m\to\infty} Q_{(m)l}(0)\Phi_{(m)l}(\zeta, \theta) = Q_{(\infty)l}(0)\Phi_{(\infty)l}(\zeta, \theta) = q_0 \Phi_{(\infty)l}(\zeta, \theta) \tag{40}$$

with corresponding equations for the P and J operators. This result extends to any finite polynomial $A(Q_{(m)l}(0), P_{(m)l}(0), J_{(m)l}(0))$. The vectors

$$\Phi_{(m)l}(\zeta, \theta) \qquad A(Q_{(m)l}(0), P_{(m)l}(0), J_{(m)l}(0))\Phi_{(m)l}(\zeta, \theta) \tag{41}$$

are associated with elements of $\mathcal{T}_{(m)l}$, while the vectors

$$\Phi_{(\infty)l}(\zeta, \theta) \qquad A(Q_{(\infty)l}(0), P_{(\infty)l}(0), J_{(\infty)l}(0))\Phi_{(\infty)l}(\zeta, \theta) \tag{42}$$

are associated with elements of $\mathcal{T}_{(\infty)l}$.

The next step needed in the argument is to show that (40) can be extended to times $t > 0$. This is difficult to establish for general polynomial Hamiltonians, however it should be emphasized that it is a quite straightforward matter to establish the result when the operators $Q_{(m)l}(t)$, $P_{(m)l}(t)$, $J_{(m)l}(t)$ are polynomial in $Q_{(m)l}(0)$,

$P_{(m)l}(0)$, $J_{(m)l}(0)$ and analytic in t. Let $Q_{(m)l}(t)$, $P_{(m)l}(t)$, $J_{(m)l}(t)$ be the solution of the operator differential equations (36) with the initial condition $Q_{(m)l}(0)$, $P_{(m)l}(0)$, $J_{(m)l}(0)$ as given in (33). Due to the finite dimension of the space $S_{(m)l}$, these solutions are known to exist and to be analytic in t for all t, from (38). For r, s, $-M_{(m)l} \leqslant r, s \leqslant 2ml - M_{(m)l}$, let

$$[Q_{(m)l}(t)]_{rs} = (\psi_{(m)r}, Q_{(m)l}(t)\psi_{(m)s})_{(m)l} \tag{43}$$

denote the matrix elements of $Q_{(m)l}(t)$ in the $\psi_{(m)r}$ basis, and similarly define $[P_{(m)l}(t)]_{rs}$, $[J_{(m)l}(t)]_{rs}$. Define $Q_{(\infty)l}(t)$ in terms of the limits

$$\begin{aligned} [Q_{(\infty)l}(t)]_{rs} &= (\psi_{(\infty)r}, Q_{(\infty)l}(t)\psi_{(\infty)s})_{(\infty)l} \\ &= \lim_{m\to\infty} (\psi_{(m)r}, Q_{(m)l}(t)\psi_{(m)s})_{(m)l} \\ &= \lim_{m\to\infty} [Q_{(m)l}(t)]_{rs} \end{aligned} \tag{44}$$

with corresponding equations for $[P_{(\infty)l}(t)]_{rs}$, $[J_{(\infty)l}(t)]_{rs}$, meaning pointwise convergence in t for all integers r, s. In these terms, the limit as $m \to \infty$ of the evolution operator $U_{(m)l}(t)$ does not exist.

It is conjectured that the connection between the quantum and classical dynamics is then established through the extension of (40) to times $t > 0$, so that

$$\lim_{m\to\infty} Q_{(m)l}(t)\Phi_{(m)l}(\zeta,\theta) = Q_{(\infty)l}(t)\Phi_{(\infty)l}(\zeta,\theta) = q(t)\Phi_{(\infty)l}(\zeta,\theta) \tag{45}$$

with corresponding limits for the P, J operators, where $(q(t), p(t), u(t))$ is the solution of (31) having the initial condition (q_0, p_0, u_0). Then the generalized one-dimensional subspace spanned by $\Phi_{(\infty)l}(\zeta,\theta)$ will be invariant under the action of $Q_{(\infty)l}(t)$, $P_{(\infty)l}(t)$ and $J_{(\infty)l}(t)$ in the contraction limit, and will be an eigenspace of those operators, corresponding to eigenvalues on a compact classical trajectory.

Key steps in this argument that have yet to be established for general Hamiltonians involve showing that the limits (44) exist and that the results (45) follow. In essence though, this again amounts to demonstrating the interchangeability of two limiting processes.

Rather than exploring these difficult problems for general Hamiltonians at this stage [7], simple systems for which the idea can be pushed through completely are considered as support for the conjecture.

3.1. Systems with simple dynamics

As noted earlier, it is easy to establish the desired result (45) for Hamiltonians (27) which lead to the dynamics being polynomial in $Q_{(m)l}(0)$, $P_{(m)l}(0)$, $J_{(m)l}(0)$ and analytic in t. For example, consider the compact so(3) oscillator with Hamiltonian

$$H_l(Q_l, P_l, J_l) = P_l^{\,2}/(2M) + \tfrac{1}{2}M\omega^2 Q_l^{\,2} \tag{46}$$

where M is the mass of the oscillator, ω is the angular frequency of oscillation, and the length and momentum scales λ, κ satisfy

$$\lambda = \sqrt{\hbar/M\omega} \qquad \kappa = \sqrt{\hbar M\omega}. \tag{47}$$

A case where the length and momentum scales chosen differ from (47) will be considered in section 3.2. The analogous compact classical Hamiltonian function is

$$H(q,p,u) = p^2/(2M) + \tfrac{1}{2}M\omega^2 q^2 \tag{48}$$

and the compact classical dynamical equations are, from (40),

$$\dot{q} = pu/M \qquad \dot{p} = -M\omega^2 uq \qquad \dot{u} \equiv 0 \tag{49}$$

with general solutions of the form

$$\begin{aligned} q(t) &= q_0\cos(\omega u_0 t) + (p_0/M\omega)\sin(\omega u_0 t) \\ p(t) &= p_0\cos(\omega u_0 t) - M\omega q_0\sin(\omega u_0 t) \\ u(t) &\equiv u_0 \end{aligned} \tag{50}$$

where $(q_0, p_0, u_0) = (q(0), p(0), u(0))$ is the initial condition of the compact classical trajectory, and

$$[{q_0}^2/\lambda^2 + {p_0}^2/\kappa^2]/l + {u_0}^2 = 1. \tag{51}$$

For each m,

$$\begin{aligned} H_{(m)l}(Q_{(m)l}, P_{(m)l}) &= {P_{(m)l}}^2/2M + \tfrac{1}{2}M\omega^2 {Q_{(m)l}}^2 \\ &= \hbar\omega({P_{(m)l}}^2/2\kappa^2 + {Q_{(m)l}}^2/2\lambda^2). \end{aligned} \tag{52}$$

Consequently,

$$\begin{aligned} \dot{Q}_{(m)l} &= (1/M)(P_{(m)l}J_{(m)l} - (\mathrm{i}\epsilon_m M\omega/2l)Q_{(m)l}) \\ \dot{P}_{(m)l} &= -M\omega^2(Q_{(m)l}J_{(m)l} + (\mathrm{i}\epsilon_m/2lM\omega)P_{(m)l}) \\ \dot{J}_{(m)l} &\equiv 0 \end{aligned} \tag{53}$$

where ϵ_m is the contraction parameter. The equation $\dot{J}_{(m)l} \equiv 0$ stems from (47) and this also results in the Hamiltonian operator $H_{(m)l}$ being diagonal on the basis vectors $\psi_{(m)r}$. Solutions of (53) can be found in the form

$$\begin{aligned} Q_{(m)l}(t) \pm \frac{\mathrm{i}}{M\omega}P_{(m)l}(t) = &\left[Q_{(m)l}(0) \pm \frac{\mathrm{i}}{M\omega}P_{(m)l}(0)\right] \\ &\times \exp[(\mp\mathrm{i}\omega J_{(m)l}(0) - \mathrm{i}\epsilon_m\omega\mathbb{I}_{(m)l}/2l)t]. \end{aligned} \tag{54}$$

Now as $m \to \infty$,

$$\exp[(\mp\mathrm{i}\omega J_{(m)l}(0) - \mathrm{i}\epsilon_m\omega\mathbb{I}_{(m)l}/2l)t] \to \exp[\mp\mathrm{i}\omega u_0 t]\mathbb{I}_{(\infty)l} \tag{55}$$

and

$$Q_{(\infty)l}(t) \pm \frac{\mathrm{i}}{M\omega}P_{(\infty)l}(t) = \left[Q_{(\infty)l}(0) \pm \frac{\mathrm{i}}{M\omega}P_{(\infty)l}(0)\right]\exp[\mp\mathrm{i}\omega u_0 t] \tag{56}$$

with

$$J_{(\infty)l}(t) \equiv J_{(\infty)l}(0). \tag{57}$$

Thus

$$\begin{aligned} Q_{(\infty)l}(t) &= Q_{(\infty)l}(0)\cos[\omega u_0 t] + (1/M\omega)\sin[\omega u_0 t] \\ P_{(\infty)l}(t) &= P_{(\infty)l}(0)\cos[\omega u_0 t] - M\omega\sin[\omega u_0 t] \end{aligned} \tag{58}$$

and clearly

$$Q_{(\infty)l}(t)\Phi_{(\infty)l}(\zeta,\theta) = q(t)\Phi_{(\infty)l}(\zeta,\theta) \tag{59}$$

with corresponding equations for $P_{(\infty)l}(t)$ and $J_{(\infty)l}(t)$. While $\exp[(\mp i\omega J_{(m)l}(0) - i\epsilon_m \omega \mathbb{I}_{(m)l}/2l)t]$ is not an element of the enveloping algebra of $Q_{(m)l}(0)$, $P_{(m)l}(0)$, $J_{(m)l}(0)$, its behaviour in the contraction limit can nonetheless be determined.

Another simple example where the idea can be pushed through completely has the Hamiltonian

$$H_{(m)l}(Q_{(m)l}, P_{(m)l}, J_{(m)l}) = J_{(m)l} \tag{60}$$

and solutions

$$\begin{aligned} Q_{(m)l}(t) &= Q_{(m)l}(0)\cos\omega t - (\lambda/\kappa)P_{(m)l}(0)\sin\omega t \\ P_{(m)l}(t) &= P_{(m)l}(0)\cos\omega t + (\kappa/\lambda)Q_{(m)l}(0)\cos\omega t \\ J_{(m)l}(t) &\boxminus J_{(m)l}(0) \end{aligned} \tag{61}$$

where $\omega = 1/l\lambda\kappa$.

3.2. A numerical example

The presence of the Dirac delta function in (25) is not helpful in terms of numerical treatment of the contraction process. Therefore, an alternative strategy is employed. This involves normalized truncations of the vectors $\Phi_{(m)l}(\zeta,\theta)$ which weakly converge to the zero vector of $\mathcal{H}_{(\infty)l}$ in the contraction limit. The sequence of normalized vectors is constructed as follows. Set $M'_{(m)l} = [\![\zeta m^\gamma]\!] \leqslant M_{(m)l}$ in the case $0 < \zeta \leqslant l$; and $M'_{(m)l} = [\![(2l-\zeta)m^\gamma]\!] \leqslant M_{(m)l}$ in the case $l < \zeta < 2l$, where in both cases $0 < \gamma < 1$. Define

$$\phi_{(m)l}(\zeta,\theta) = \frac{1}{\sqrt{2M'_{(m)l}+1}} \sum_{r=-M'_{(m)l}}^{M'_{(m)l}} e^{ir\theta}\psi_{(m)r} \tag{62}$$

The constant γ is a dimensionless parameter introduced into the construction of $\phi_{(m)l}(\zeta,\theta)$ to obtain (63) and (64). Because

$$\lim_{m\to\infty}(M'_{(m)l}/M_{(m)l}) = 0 \tag{63}$$

it can be proved inductively that

$$\lim_{m\to\infty}(\phi_{(m)l}(\zeta,\theta), A(Q_{(m)l}(0), P_{(m)l}(0), J_{(m)l}(0))\phi_{(m)l}(\zeta,\theta))_{(m)l}$$
$$= A(q_0, p_0, u_0) \tag{64}$$

for any polynomial $A(Q,P,J)$, and so it is conjectured that (64) holds for times $t > 0$ and that the spread of the wavepacket will vanish as $m \to \infty$. This conjecture has been investigated numerically for a system whose compact classical behaviour is nonlinear but integrable. Consider the Hamiltonian operator

$$H_l(Q_l, P_l, J_l) = \tfrac{1}{2}(P_l{}^2/M + 2M\omega^2 Q_l{}^2) \tag{65}$$

with λ, κ again satisfying (47). The corresponding compact classical system has Hamiltonian

$$H(q,p,u) = \tfrac{1}{2}(p^2/M + 2M\omega^2 q^2) \tag{66}$$

with dynamical equations

$$\dot{q} = (1/m)pu \qquad \dot{p} = -(2m\omega^2)qu \qquad \dot{u} = (\omega/l\hbar)qp. \tag{67}$$

This system has oscillatory solutions [7, 9] given in terms of the Jacobi elliptic functions $\mathrm{sn}(\cdot)$, $\mathrm{cn}(\cdot)$, $\mathrm{dn}(\cdot)$ [11], where the frequency of oscillation and the form of the solutions is amplitude-dependent.

For each m, define

$$H_{(m)l}(Q_{(m)l}, P_{(m)l}, J_{(m)l}) = \hbar\omega(P_{(m)l}{}^2/2\kappa^2 + 2Q_{(m)l}{}^2/2\lambda^2) \tag{68}$$

and consequently, $H_{(m)l}$ is now not diagonal on the basis vectors $\psi_{(m)r}$.

The initial conditions for the compact classical trajectory chosen for the numerical study are $(q_0/\lambda, p_0/\kappa) = (0.5, 0.0)$ for the position and momentum with u_0 being the positive value determined from (51). In each figure, non-dimensional time ωt is plotted along the horizontal axis and non-dimensional position q/λ is plotted along the vertical axis. The line $q = 0$ is shown. The other three lines show, for various values of m, the non-dimensional expectation value $(1/\lambda)\langle Q_{(m)l}(t)\rangle$ and its non-dimensional spread $(1/\lambda)(\langle Q_{(m)l}(t)\rangle \pm \sigma_{Q_{(m)l}(t)})$. The fourth line shows the periodic (non-sinusoidal) compact classical trajectory with the initial conditions given earlier. For the purposes of these computations, $M'_{(m)l}$ was taken to be $\min([\![\sqrt{m}]\!], M_{(m)l})$; that is $\gamma = 0.5$, the deletion of ζ from the definition of $M'_{(m)l}$ not changing the essential idea. Also, $l = 1$ was chosen, giving $\zeta \approx 0.133\,97$ and $\theta = 0$. The operator $Q_{(m)l}(t)$ was computed numerically over a range of t values by diagonalizing the evolution operator $U_{(m)l}(t)$ and using (38).

Figure 1 shows the non-dimensional position expectation value and spread for the compact oscillator with $m = 49.5$, $M'_{(m)l} = 6$, $M_{(m)l} = 6$ and the dimension of the representation being $2ml + 1 = 100$.

Figure 2 shows the non-dimensional position expectation value and spread for $m = 299.5$, $M'_{(m)l} = 17$, $M_{(m)l} = 40$ and the dimension of the representation being 600.

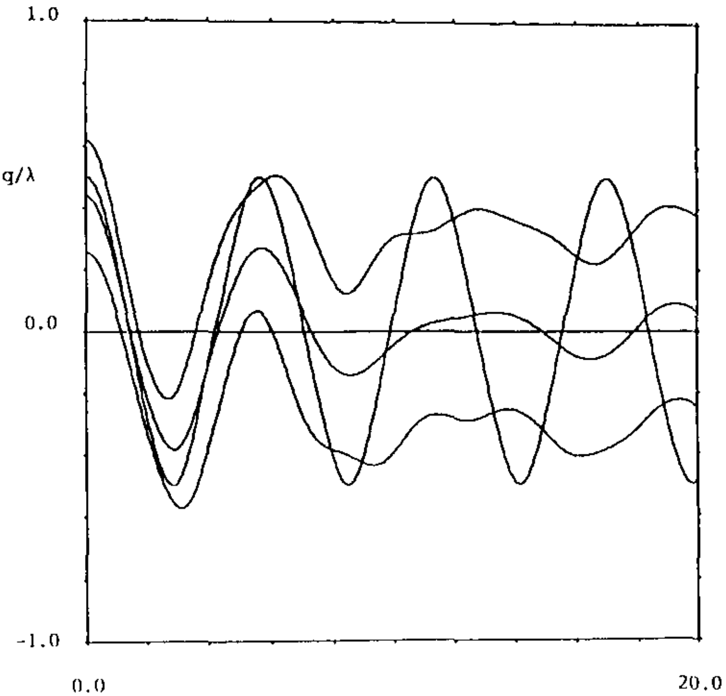

Figure 1. Spread of the so(3) quantum wavepacket with respect to position for $l = 1$ and $m = 49.5$.

Figure 3 shows the non-dimensional position expectation value and spread for $m = 299.5$, $M'_{(m)l} = 17$, $M_{(m)l} = 119$ with a new starting value of $(q_0/\lambda, p_0/\kappa) = (0.8, 0)$ for the compact classical trajectory, with u_0 again being the positive value determined by (51). The dimension of the representation is 600 and $\zeta = 0.4$, $\theta = 0$. The compact classical behaviour of the position variable is still periodic but confined to positive values.

Figures 1 and 2 show how the spread of the wavepacket decreases as m increases and how the expectation values follow more closely the compact classical trajectory. Figure 3 shows that the compact classical behaviour for an extreme initial position is also recovered [7,9].

3.3. The so(3) compact classical Poisson bracket

For systems where the compact quantum dynamics depends polynomially on $Q_l(0)$, $P_l(0)$, $J_l(0)$ and analytically on t, the corresponding compact classical systems will exhibit analogous dependencies. For such systems, it can therefore be established directly, along similar lines to the non-compact case [1], that

$$\begin{aligned}\lim_{m\to\infty} Q_{(m)l}(t)\Phi_{(m)l}(\zeta,\theta) &= Q_{(\infty)l}(t)\Phi_{(\infty)l}(\zeta,\theta)\\ &= q(t)\Phi_{(\infty)l}(\zeta,\theta)\end{aligned} \tag{69}$$

with corresponding limits for the $P_{(m)l}(t)$, $J_{(m)l}(t)$ operators, so that each one-dimensional generalized subspace of $\mathcal{H}_{(\infty)l}$ spanned by $\Phi_{(\infty)l}(\zeta,\theta)$ remains invariant

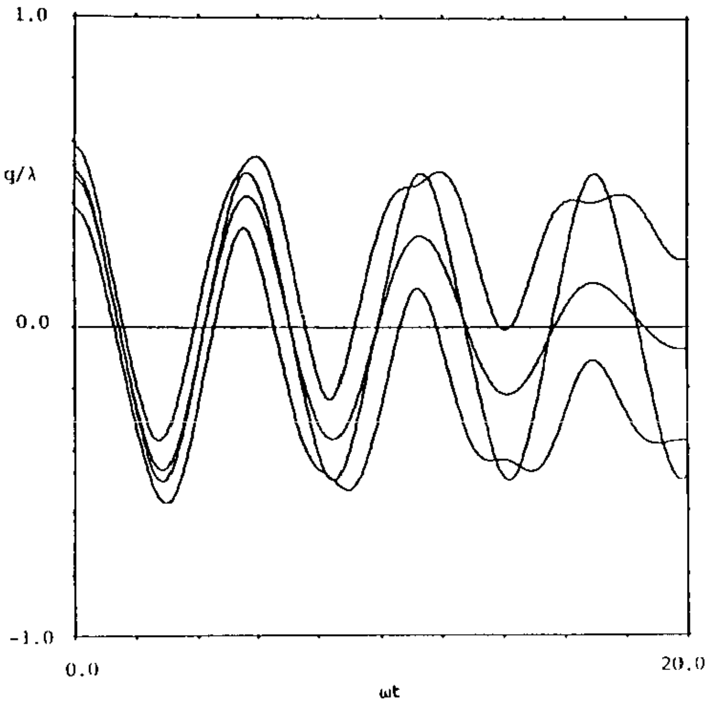

Figure 2. Spread of the so(3) quantum wavepacket with respect to position for $l = 1$ and $m = 299.5$.

under the action of these operators, and is an eigenspace of $(Q_{(\infty)l}(t), P_{(\infty)l}(t), J_{(\infty)l}(t))$ corresponding to eigenvalues on a compact classical trajectory. This is automatically extended to

$$\begin{aligned}\lim_{m\to\infty} & A(Q_{(m)l}(t), P_{(m)l}(t), J_{(m)l}(t))\Phi_{(m)l}(\zeta,\theta) \\ &= A(Q_{(\infty)l}(t), P_{(\infty)l}(t), P_{(\infty)l}(t))\Phi_{(\infty)l}(\zeta,\theta) \\ &= A(q(t), p(t), u(t))\Phi_{(\infty)l}(\zeta,\theta) \end{aligned} \tag{70}$$

for any polynomial $A(Q, P, J)$, including the Hamiltonian operator $H(Q, P, J)$. It can also be seen that for each example,

$$\begin{aligned}\lim_{m\to\infty} \dot{Q}_{(m)l}(t)\Phi_{(m)l}(\zeta,\theta) &= \lim_{m\to\infty} \frac{1}{\mathrm{i}\epsilon_m \hbar}[Q_{(m)l}(t), H_{(m)l}]\Phi_{(m)l}(\zeta,\theta) \\ &= \dot{Q}_{(\infty)l}(t)\Phi_{(\infty)l}(\zeta,\theta) \\ &= \{q, H(q,p,u)\}_{\mathrm{so}(3)}\Phi_{(\infty)l}(\zeta,\theta) \end{aligned} \tag{71}$$

with corresponding limits for $\dot{P}_{(m)l}(t)$, $\dot{J}_{(m)l}(t)$, where

$$\begin{aligned}\{A, B\}_{\mathrm{so}(3)} = \frac{1}{l\lambda^2} q \left(\frac{\partial A}{\partial p}\frac{\partial B}{\partial u} - \frac{\partial A}{\partial u}\frac{\partial B}{\partial p}\right) + \frac{1}{l\kappa^2} p \left(\frac{\partial A}{\partial u}\frac{\partial B}{\partial q} - \frac{\partial A}{\partial q}\frac{\partial B}{\partial u}\right) \\ + u \left(\frac{\partial A}{\partial q}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial q}\right) \end{aligned} \tag{72}$$

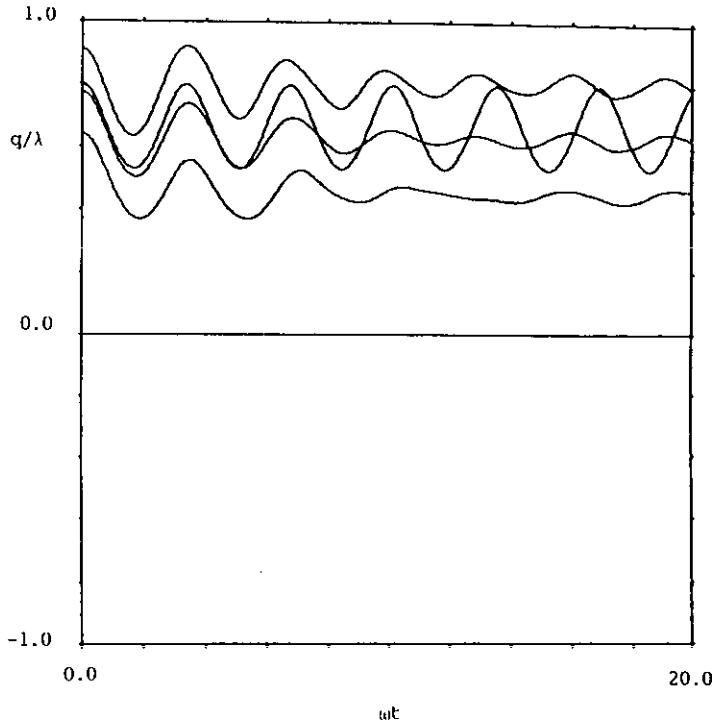

Figure 3. Spread of an alternative so(3) quantum wavepacket with respect to position for $l = 1$ and $m = 299.5$.

is the compact classical Poisson bracket for so(3) compact classical systems [8,9]. In terms of this Poisson bracket, which has a cyclic symmetry in the three variables, (31) can be written in the form

$$\dot{q} = \{q, H\}_{\mathrm{so}(3)} \qquad \dot{p} = \{p, H\}_{\mathrm{so}(3)} \qquad \dot{u} = \{u, H\}_{\mathrm{so}(3)}. \tag{73}$$

4. The relationship to the non-compact result

Throughout this analysis, the parameter l has been treated as being constant. However, in [6], $\epsilon_l = l^{-1/2}$ was treated as a contraction parameter. By sending l to infinity, $(\epsilon_l \to 0)$, in (3), the contraction of so(3) to w_1 is obtained, and (6) reduces to

$$\begin{aligned} Q_\infty \xi_r &= (\lambda/\sqrt{2})[\sqrt{r}\xi_{r-1} + \sqrt{r+1}\xi_{r+1}] \\ P_\infty \xi_r &= (-\mathrm{i}\kappa/\sqrt{2})[\sqrt{r}\xi_{r-1} - \sqrt{r+1}\xi_{r+1}] \\ J_\infty \xi_r &= \xi_r \end{aligned} \tag{74}$$

which defines matrix elements for the familiar Hermitian matrix representation of w_1.

Keeping m fixed where appropriate, it is possible to send l to infinity in the various equations given earlier and recover the appropriate equations from the analogous discussion of obtaining the classical limit of a w_1 non-compact quantum system by means of the contraction of w_1 (and its representations) to t_3 (and its representations).

Further, by sending l to infinity in (31) and (32), $\dot{u} \to 0$, $|u| \to 1$, and if $H(q,p,u)$ does not depend explicitly on u, then (31) reduces to the familiar equations of Hamiltonian mechanics in one spatial variable

$$\dot{q} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial q}. \tag{75}$$

Likewise, the so(3) Poisson bracket (72) reduces to the familiar non-compact Poisson bracket

$$\{A,B\} = \frac{\partial A}{\partial q}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial q}. \tag{76}$$

The motion on the compact bounded ellipsoid of revolution (32) flattens out onto one of a pair of parallel planes corresponding to $|u| = 1$.

Thus there is a dual contraction process:

$$\begin{array}{ccc} \text{so}(3) \text{ quantum systems} & \xrightarrow{\epsilon_l} & w_1 \text{ quantum systems} \\ \epsilon_m \downarrow & & \downarrow \epsilon_m \\ \text{so}(3) \text{ classical systems} & \xrightarrow{\epsilon_l} & w_1 \text{ classical systems} \end{array}$$

where the contractions may be done in either order.

5. Concluding remarks

The classical limit of $\text{so}(n+2)$ compact quantum systems of the type discussed in [6] with n pairs of compact position and momentum operators can be considered along the lines presented here.

However the generalization of the so(3) formulation to the $\text{so}(n+2)$ formulation is not quite as straightforward as the generalization of the w_1 formulation to the w_n formulation. This is mostly due to the difficulties involved in writing down explicitly matrix elements of representatives of the generators for general $(l, l_1, \ldots, l_{n'})$, $n' = [\![n/2]\!]$ irreducible Hermitian matrix representations of $\text{so}(n+2)$ with the last generator $J_{n+1\,n+2}$ being diagonal *and* with its (degenerate) eigenvalues arranged in ascending or descending order when $n > 2$. The $\text{so}(n+2)$ compact quantum system has operators [6]

$$\begin{aligned} &J_{rs} && r,s = 1,\ldots,n \ (r \neq s) \\ &Q_r = (\kappa/\sqrt{l})J_{r\,n+1} && r = 1,\ldots,n \\ &P_r = (-\kappa/\sqrt{l})J_{r\,n+2} && r = 1,\ldots,n \\ &J = -(1/l)J_{n+1\,n+2} \end{aligned} \tag{77}$$

and the desired contraction to $t_{(n+1)(n+2)/2}$ is achieved by setting $J^{\epsilon}_{rs} = \epsilon J_{rs}$, $Q^{\epsilon}_r = \epsilon Q_r$, $P^{\epsilon}_r = \epsilon P_r$, $J^{\epsilon} = \epsilon J$. A possible choice of sequence of representations would be those labelled $(lm, l_1, l_2, \ldots, l_{n'})$ for $m = 1, 2, \ldots$, with $\epsilon_m = 1/m$. The quadratic Casimir operator of so$(n+2)$ again plays an important part, inducing a positive definite quadratic constraint involving all the compact classical variables and restricting the compact classical motion to a compact hyperellipsoid in the $(n+1)(n+2)/2$ dimensional classical phase space.

It is conjectured that similar results to the examples considered here will hold for so(3) systems with arbitrary polynomial Hamiltonians $H_l(Q_l, P_l, J_l)$. Clearly general technical conditions need to be investigated under which (69) is satisfied. Such conditions, however, would not need to be quite so severe as those for the non-compact case, due to the known existence of analytic solutions $Q_{(m)l}(t)$, $P_{(m)l}(t)$, $J_{(m)l}(t)$ for arbitrary polynomial Hamiltonians. It is hoped that this will be the topic for further studies.

It should be noted in closing that the remarks reported in the non-compact case concerning superselection rules are also valid here: the Hilbert space $\mathcal{H}_{(\infty)l}$ is decomposable into a direct integral of generalized one-dimensional subspaces, each of which remains invariant under the dynamics. Each of these subspaces is associated with a single compact classical trajectory and the superposition of vectors from two or more of these subspaces is unphysical.

Acknowledgments

This work derives from a PhD thesis [7]. I am particularly endebted to Professor A J Bracken for many helpful discussions. The financial support of an Australian Commonwealth Postgraduate Research Award is gratefully acknowledged.

References

[1] Fawcett R J B and Bracken A J 1991 *J. Phys. A: Math. Gen.* **24** 2743
[2] Barut A O and Bracken A J 1981 *Phys. Rev.* **23** 2454
[3] Barut A O and Bracken A J 1985 *J. Math. Phys.* **26** 2515
[4] Barut A O, Bracken A J and Thacker W D 1984 *Lett. Math. Phys.* **8** 477
[5] Barut A O 1986 *Lecture Notes in Physics* vol 261 (Berlin: Springer)
[6] Fawcett R J B and Bracken A J 1988 *J. Math. Phys.* **29** 1521
[7] Fawcett R J B 1990 *PhD Thesis* University of Queensland
[8] Grossmann Z and Peres A 1963 *Phys. Rev.* **132** 2346
[9] Fawcett R J B 1992 *J. Math. Phys.* at press
[10] Exner P, Havlicek M and Lassner W 1976 *Czech. J. Phys.* B **26** 1213
[11] Spiegel M R 1968 *Mathematical Handbook* (New York: McGraw-Hill)